Chapter I: Labour Market Theory

Section 2: Mortensen-Pissarides matching model

Literature:

Pierre Cahuc and André Zylberberg: Labour Economics
   Chapter 9: Job Reallocation and Unemployment

Christopher Pissarides: Equilibrium Unemployment
   Chapter 1: The Labor Market
2.1 Matching function

Definition: Matching function

A matching function determines the number of new hires $M(V, U)$ given the number of unemployed workers $U$ and the number of vacancies $V$.

Definition: Market tightness

The market tightness $\theta$ equals the number of vacancies per unemployed worker, i.e. $\theta = V/U$. 
2.1.1 The general matching function

Properties:

The number of new hires is zero, if the number of unemployed or the number of vacancies is zero, i.e.
\[ M(V, 0) = M(0, U) = 0. \]

The number of new hires increases with the number of unemployed and the number of vacancies, i.e.
\[ M'(V, U) > 0 \quad \text{and} \quad M'_V (V, U) > 0. \]

The matching function has constant returns to scale, i.e.
\[ M(\lambda V, \lambda U) = \lambda M(V, U). \]
Matching probability of a vacancy:

\[ \frac{M(V, U)}{V} = M(1, U/V) \equiv m(\theta), \tag{9} \]

The probability that a vacancy meets a worker decreases with market tightness \( \theta \), i.e.

\[ m'(\theta) = \frac{\partial M(1, 1/\theta)}{\partial \theta} = -\frac{1}{\theta^2} M'_U(1, 1/\theta) < 0. \]

Intuition:

If the number of vacancies (in relation to the number of unemployed) increases, then it is less likely that a vacancy is able to hire an unemployed worker (congestion externality).
Matching probability of a worker:

\[
\frac{M(V, U)}{U} = \frac{V M(V, U)}{U} \equiv \theta_m(\theta). \tag{10}
\]

The probability that a worker meets a vacancy increases with market tightness \(\theta\), i.e.

\[
[\theta_m(\theta)]' = \frac{\partial M(\theta, 1)}{\partial \theta} = M'_V(V/U, 1) > 0.
\]

**Intuition:**

If the number of vacancies (in relation to the number of unemployed) increases, then it is more likely that a worker is hired by a vacancy.
2.1.2 Microeconomic foundations

A: Urnball matching

Idea: All unemployed workers send one application to a randomly selected vacancy (random search), i.e. they are unable to coordinate their applications.

1. The probability that an individual send her application to a specific vacancy is $1/V$.
2. The probability that an individual does not apply at a specific vacancy is $1 - 1/V$.
3. The probability that $U$ unemployed worker do not apply at a specific vacancy is $(1 - 1/V)^U$.
4. The probability that at least one unemployed worker applied to a specific vacancy is $1 - (1 - 1/V)^U$.
5. Each vacancy selects only one worker.
The total number of matches:

\[
M(V, U) = V \left[1 - \left(1 - \frac{1}{V}\right)^U\right].
\]  

(11)

If we let the number of vacancies and unemployed workers go to infinity, and keep the market tightness \(\theta = V/U\) constant, then the total number of matches equals

\[
M(V, U) = V \left[1 - \exp(-U/V)\right]
\]

or

\[
m(\theta) = 1 - e^{-1/\theta}
\]

(12)  

(13)

The urnball-matching function has constant returns to scale, i.e.

\[
M(\lambda V, \lambda U) = \lambda M(V, U)
\]

= \lambda M(V, U)
Multiple applications:

If unemployed workers send $a > 1$ instead of only one application, then the **number of vacancies** receiving **at least one application** depends on the wages offered.

- **If all firms offer the same wage**, workers randomize and the number of vacancies with at least one application is
  \[
  V \left[ 1 - \left( 1 - \frac{a}{V} \right)^U \right].
  \]

- **If firms offer different wages and workers send one application to each kind of firm**, the number of vacancies with at least one application is
  \[
  \sum_{i=1}^{a} V_i \left[ 1 - \left( 1 - \frac{1}{V_i} \right)^U \right] \quad \text{with} \quad \sum_{i=1}^{a} V_i = V.
  \]
In general, not all firms with at least one application will be able to hire a worker, since an unemployed worker has other competing offers.

The **number of matches** depends on the **wage mechanism** (Gautier and Holzner, 2011):

- If firms post fixed wages and commit not to increase their initial wage offers, the number of matches will be below the maximum matchings possible.

- Ex-post competition achieves the maximum matchings.
B: Stock-Flow matching (Coles and Smith, 1998)

Idea:

- Workers and firms are heterogenous, i.e. not all workers are suitable for all jobs.
- Workers look for a job on the web. This implies that all workers can contact all firms. Unemployment occurs because it takes time to process applications.

Flows match with stocks:

- Workers that became unemployed recently, $u$, check the whole stock of vacancies on the local matching website.
- Newly opened vacancies, $v$, check the whole stock of unemployed on the local matching website.

Stocks match with flows:

- Long-term unemployed, $U$, check the newly opened vacancies on the local matching website.
- Old vacancies, $V$, check the short-term unemployed on the local matching website.
Deriving the matching function

- The probability that a short-term unemployed is suitable for a specific vacancy is $\alpha$.

- The probability that a short-term unemployed does not apply to a specific vacancy is $1 - \alpha$.

- The probability that a short-term unemployed does not apply to any of the $V$ vacancies in the stock is $(1 - \alpha)^V$.

- The probability that a short-term unemployed applies to at least one vacancy is $1 - (1 - \alpha)^V$.

- This equals the matching probability of a short-term unemployed.

- If a short-term unemployed does not accept a job offer, she becomes long-term unemployed in the next period, i.e. enters the stock $U$. 
- The probability that a new vacancy does not get any application for a short-term unemployed is $1 - \alpha$.

- The probability that none of the long-term unemployed in the stock $U$ applies to a specific new vacancy is $(1 - \alpha)^U$.

- The probability that a new vacancy got at least one application for a long-term unemployed is $1 - (1 - \alpha)^U$.

- The number of matches for a given inflow $u$ of short-term unemployed and $v$ of new vacancies is

$$M (V, v, U, u) = u \left[ 1 - (1 - \alpha)^V \right] + v \left[ 1 - (1 - \alpha)^U \right].$$
Properties of the matching function:

The matching function as increasing returns to scale, since \((1 - \alpha)X^\lambda < (1 - \alpha)^X\) for \(X > 1\), i.e.

\[
M (\lambda V, \lambda v, \lambda U, \lambda u) = \\
= \lambda M (V, v, U, u)
\]

Increasing returns to scale imply that the matching probability increases with the size of the labour market.

Empirical relevance:

The stock-flow matching function is empirically supported by the fact that the stock of vacancies has no influence on the job finding rate of long-term unemployed, while the number of new vacancies has. In addition, the job finding rate of short-term unemployed is influenced by both the stock and the inflow of new vacancies.
2.2 Worker behaviour

Assumptions:

- Unemployment income (value of leisure) \( z \) per period.
- The matching probability of unemployed workers is given by \( \theta m(\theta) \).
- The expected, discounted life-time utility of an unemployed worker is given by \( V_u \).
- Workers are risk neutral and consume their per period wage \( w \), i.e. no savings.
- \( V_e(w) \) the expected, discounted life-time utility of a worker employed at wage \( w \).
- Employment ends at rate \( q \) (job destruction rate).
Bellman equations:

Derivation of the Bellman equations for unemployed and employed workers:

The **expected, discounted life-time utility**:

\[
V_t = \sum_{i=t}^{\infty} \frac{x}{(1 + r)^{i-t}}
\]

where

\[
x = \frac{z}{1 + r}, \text{ if } V_t = V_u \text{ and }
x = \frac{w}{1 + r}, \text{ if } V_t = V_e.
\]

The payments are discounted, since they are paid at the end of a period.

**Rewrite** the expected, discounted life-time utility:

\[
V_t = x + \sum_{i=t+1}^{\infty} \frac{x}{(1 + r)^{i-t+1}} = x + \frac{E_t [V_{t+1}]}{1 + r}.
\]
Bellman equation for unemployed workers:

- Substitute $x = \frac{z}{1 + r}$
- The expected life-time utility next period depends on the matching probability of a worker, i.e.
  \[ E_t[V_{t+1}] = V_u + \theta m(\theta) \max [V_e(w) - V_u, 0] \]
  and the value of employment $V_e(w)$.

An unemployed worker only accepts a job, if the value of employment is at least as high as the value of unemployed, i.e.
  \[ V_e(w) \geq V_u. \]

A worker’s **reservation wage** is defined by
  \[ V_e(w^r) = V_u. \]
For wages above the reservation wage, i.e. $w \geq w^r$, the expected life-time utility of an unemployed worker is given by

$$V_u = \frac{1}{1+r} \left[ z + V_u + \theta m(\theta) [V_e(w) - V_u] \right].$$

Rearranging implies the Bellman equation for an unemployed worker, i.e.

$$rV_u = z + \theta m(\theta) [V_e(w) - V_u],$$

(14)
Bellman equation for employed workers:

- Substitute $x = \frac{w}{1 + r}$
- The expected life-time utility next period depends on the matching probability of a worker, i.e.

$$E_t \left[ V_{t+1} \right] = V_e(w) + q \left[ V_u - V_e(w) \right].$$

Note, that the worker cannot decide whether she becomes unemployed or not.

Rearranging implies the **Bellman equation for an employed worker**, i.e.

$$rV_e(w) = w + q \left[ V_u - V_e(w) \right]. \quad (15)$$
Gains from employment:

Using equations (15) the gains form finding a job \( V_e(w) - V_u \) is given by

\[
V_e(w) - V_u = \frac{w - rV_u}{r + q}.
\] (16)

Interpretation:

- Gains from employment only occur, if the wage \( w \) is above the flow-value of being unemployed \( rV_u \).
- The gains from employment depend not only on the wage \( w \), but also on the market tightness \( \theta \), since the value of being unemployed increases with the market tightness (see equation (14)).
Reservation wage:

Equation (16) implies that the reservation wage equals unemployment benefits

\[
\begin{align*}
w^r &= rV_u \\
      &= z + \theta m(\theta) [V_e(w^r) - V_u] \\
      &= z
\end{align*}
\]

Intuition:

Workers are only willing to work for a wage higher than unemployment benefits (value of leisure).
2.3 Firm behaviour

Assumptions:

- $h$ cost of vacancy creation.
- $m(\theta)$ probability of contacting an unemployed worker.
- $\Pi_v$ expected, discounted payoff of a vacancy.
- $y$ worker-firm pairs productivity, with $y > z$.
- $\Pi_e(w)$ expected, discounted profit of employing a worker at wage $w$.
- $q$ job destruction rate.
Bellman equation for a vacancy:

The flow-value of a vacancy equals

\[ r \Pi_v = -h + m(\theta) \left[ \Pi_e(w) - \Pi_v \right], \quad (17) \]

- the cost of maintaining the vacancy, i.e. \(-h\),
- the expected gain of employing a worker, i.e. \(m(\theta) \left[ \Pi_e(w) - \Pi_v \right]\).

Firms’ decision:

Firms decide whether to create a vacancy given the expected wage \(w\) and the expected market tightness \(\theta\).
Free market entry:

- Firms will create vacancies as long as there is some profit from creating a new vacancy, i.e. as long as $\Pi_v \geq 0$.

- The additional vacancies increase the number of vacancies and hence the market tightness $\theta$.

- A higher market tightness decreases the probability to contact a worker, i.e. $m'(\theta) < 0$.

- The expected gain of employing a worker $m(\theta) [\Pi_e(w) - \Pi_v]$ decreases.

- This reduces the expected value of opening a vacancy until there are no profits from creating new vacancies, i.e.

$$\Pi_v = 0 \iff \frac{h}{m(\theta)} = \Pi_e(w). \quad (18)$$
Bellman equation for employing a worker:

The flow-value of employing a worker at the wage $w$ equals

$$r \Pi_e (w) = y - w + q [\Pi_v - \Pi_e (w)].$$

(19)

- the profit per period $y - w$,
- expected loss in case of job destruction, i.e. $q [\Pi_v - \Pi_e (w)]$

Note, if the job is destroyed, the firm can open a new vacancy.
Gains from employing a worker:

Using equation (19) the gain from employing a worker equals

\[ \Pi_e(w) - \Pi_v = \frac{y - w - r \Pi_v}{r + q}. \]

The free entry condition (18) implies that firms will only create vacancies, if they make a positive profit, i.e.

\[ \Pi_e(w) = \frac{y - w}{r + q} > 0, \]

i.e. if the wage is below the marginal product \( y > w \).
Vacancy creation curve:
Firms create vacancies until the cost of recruiting a worker, i.e.

\[
\frac{hV}{M(V,U)} = \frac{h}{m(\theta)},
\]
equals the expected discounted profit of employing a worker, i.e.

\[
\Pi_e(w) = \frac{y - w}{r + q} = \frac{h}{m(\theta)}.
\] (21)

The vacancy creation curve implies that the market tightness \( \theta \) decreases, if wages \( w \) increase, i.e.

\[
\frac{d\theta}{dw} = \frac{m(\theta)^2}{m'(\theta) h(r + q)} < 0.
\]
Intuition for the vacancy creation curve:

A wage increase reduces firms’ profits.
Since the creation of vacancies is costly, firms create less vacancies.
This decreases the market tightness and increases the matching probability of firms,
until the cost of recruiting a worker equals again the profit of employing a worker.

Boundaries of the vacancy creation curve:

At $w = 0$, the market tightness is at its maximum value $\bar{\theta}$, i.e. $y/(r + q) = h/m(\bar{\theta})$.
At $w = y$, profits are zero and no vacancies are created, i.e. $\theta \to 0$ as $w \to y$. 
Figure 2.1: Vacancy creation curve
2.4 Wage bargaining

Assumptions:

- If a worker and a firm meet, they have temporarily no competitors (decentralized wage determination).

- Workers and firms bargain over the match surplus $S$. We assume generalized Nash Bargaining.

- The match surplus $S$ equals the sum of the gains from employment, $V_e(w) - V_u$, and the gains from employing a worker, $\Pi_e(w) - \Pi_v$, i.e.

$$S = [V_e(w) - V_u] + [\Pi_e(w) - \Pi_v]$$

- The relative bargaining power is $\gamma \in (0, 1)$ for unemployed and $(1 - \gamma)$ for firms.
Nash’s 4 axioms:

1. Efficiency: The solution lies at the Pareto frontier (most important).
2. Symmetry: The solution is symmetric. This is relaxed in the generalized solution.
3. Independence of irrelevant alternatives: Preference between 2 alternative allocations only depends on the ranking of the allocations and on nothing else.
4. Invariance to monotonic transformations: The solution is independent of the used scaling.

Generalized Nash Bargaining solution:

Nash showed that there exists a unique wage that maximizes the product of firm and worker surpluses:

\[ w = \arg \max \left[ V_e (w) - V_u \right]^{\gamma} \left[ \Pi_e (w) - \Pi_v \right]^{(1-\gamma)} \]  \hspace{1cm} (22)
Splitting of the match surplus:

The generalized Nash Bargaining solution implies that the match surplus is split according to the relative bargaining power of the parties involved, i.e.

\[ V_e(w) - V_u = \gamma S \quad \text{and} \quad \Pi_e(w) - \Pi_v = (1 - \gamma) S. \]

Derivation:

Log-transformation:

FOC:

Substituting the surplus:
Game-theoretic foundation (Rubinstein, 1982):

Assumptions:

- Infinitely repeated ultimatum game (alternating wage offers, $w_F$ and $w_W$).
- Discounting between offers, where $\delta_F$ equals the firm discount factor and $\delta_W$ the worker discount factor.
- Firms start with their offer (not important).

$\implies$ Infinite repetition implies that workers and firms use the same strategies every period.

$\implies$ If a worker does not accept the firm’s wage offer in period 1, she will never accept it.
Optimal firm strategy:

\[ \Pi_e (w_F) - \Pi_v = \delta_F [\Pi_e (w_W) - \Pi_v] \]

Optimal worker strategy (if the first offer is not acceptable):

\[ V_e (w_W) - V_u = \delta_W [V_e (w_F) - V_u] \]

Using the definition of the surplus \( S = [V_e (w) - V_u] + [\Pi_e (w) - \Pi_v] \) implies the following solution:

\[ \Pi_e (w_F) - \Pi_v = \left[ 1 - \frac{1 - \delta_F}{1 - \delta_F \delta_W} \right] S \]

where

\[ \gamma = \frac{1 - \delta_F}{1 - \delta_F \delta_W} \]
Wage equation:

The bargaining solution implies \( \Pi_e(w) - \Pi_v = (1 - \gamma) S \).

Substituting the surplus and the gain from employing a worker, i.e.

\[
\Pi_e(w) = \frac{y - w}{r + q} \quad \text{and} \quad S = \frac{w - rV_u}{r + q} + \frac{y - w}{r + q} = \frac{y - rV_u}{r + q}
\]

implies the following wage equation:

\[
w = \gamma y + (1 - \gamma) rV_u = rV_u + \gamma (y - rV_u) .
\]

The wage equation implies that the wage \( w \) is always above the reservation wage \( w^r \), since the wage is above the value of being unemployed, i.e. \( w > rV_u \).

Intuition:

The wage increases with the value of being unemployed, since a high value of unemployment increases the worker’s outside option during the bargaining process.
Wage curve:

Substituting the value of being unemployed $rV_u$ implies the following wage curve:

$$w = z + \gamma (y - z + h\theta) \quad (23)$$

$$= z + \gamma (y - z) \frac{r + q + \theta m(\theta)}{r + q + \gamma \theta m(\theta)}. \quad (24)$$

The wage $w$ increases with the market tightness $\theta$, i.e.

$$\frac{\partial w}{\partial \theta} = \gamma h > 0 \quad \text{or} \quad \frac{\partial w}{\partial \theta} = \gamma (y - z) \frac{[1 - \gamma] [\theta m(\theta)]' [r + q]}{[r + q + \gamma \theta m(\theta)]^2} > 0$$
Intuition for the wage curve:

- If the market tightness $\theta$ increases, then the job finding rate of unemployed workers increases.

- This increases the value of being unemployed $rV_u$.

- This increases wages, since the value of being unemployed equals the worker’s outside option during the bargaining process.

Boundaries:

For $\theta \to 0$, i.e. no vacancies implies $w \to z + \gamma (y - z)$.

For $\theta \to \bar{\theta}$ implies $w \to z + \gamma (y - z + h\theta)$. 
Figure 2.2: Wage curve
2.5 Steady state in- and outflow into unemployment

Assumptions:

Let $N$ be the number of workers in the economy, $U$ the number of unemployed and $L$ the number of employed workers, with $N = L + U$.

$V$ denotes the number of vacancies.

$\dot{U}$ denotes the change in the number of unemployed workers over time.

The unemployment rate equals $u = U/N$ and the vacancy rate $v = V/N$.

$\Rightarrow$ labour market tightness equals $\theta = V/U = v/u$.

Employed workers enter unemployment at rate $q$ (job destruction rate).

Steady state:

In steady state the inflow into unemployment equals the outflow from unemployment, i.e. $\dot{U} = \ddot{u} = 0$. 
In- and outflows into unemployment:

The number of unemployed workers increases with the number of workers that are laid off, i.e. \( qL \), and decreases with the number of workers finding a new job, i.e. \( \theta m(\theta)U \),

\[
\dot{U} = qL - \theta m(\theta)U.
\]

Dividing by \( N \):

\[
\dot{u} = q [1 - u] - \theta m(\theta) u
\]

Steady state implies \( \dot{u} = 0 \). Thus, the steady state unemployment and employment rate is given by

\[
u = \frac{q}{q + \theta m(\theta)} \quad \text{and} \quad l \equiv 1 - u = \frac{\theta m(\theta)}{q + \theta m(\theta)}.
\]

The unemployment rate \( u \) increases with the job destruction rate \( q \) and decreases with the job finding rate \( \theta m(\theta) \) (and the market tightness).
The Beveridge curve:

**Definition:**

The Beveridge curve illustrates the linkage between the unemployment rate $u$ and the vacancy rate $v$, if both are in steady state, i.e.

$$u = \frac{q}{q + \theta m(\theta)}$$

The implicit function theorem implies:

$$\frac{du}{dv} = -\frac{q}{[q + \theta m(\theta)]^2} \left[\theta m(\theta)\right]' \frac{\partial \theta}{\partial v} = -\frac{u \left[\theta m(\theta)\right]' \frac{1}{q + \theta m(\theta) u}}{1 - u \frac{\frac{1}{q + \theta m(\theta) u}}{\left[\theta m(\theta)\right]'}} = -\frac{\left[\theta m(\theta)\right]'}{q + \theta m(\theta) - \theta \left[\theta m(\theta)\right]'} = -\frac{\left[\theta m(\theta)\right]'}{q + \theta m(\theta) - \theta m(\theta) - \theta^2 m'(\theta)} < 0$$
Intuition of the Beveridge curve:

For a high unemployment rate $u$ to be in steady state, the outflow out of unemployment must be relatively low.

The outflow out of unemployment is low, if the matching probability $\theta m(\theta)$ for unemployed workers is low.

The matching probability for unemployed workers is low, if the vacancy rate $v$ and therefore the market tightness $\theta$ are low.

Similarly, for a low steady state unemployment rate $u$. 
Figure 2.3: Beveridge curve
2.6 Steady state equilibrium

Definition:

A steady state equilibrium is defined by the follow three variables:

- the wage $w$,
- the labour market tightness $\theta$,
- and the unemployment rate $u$.

1. The wage curve and the job creation rate determine the wage $w^*$ and the labour market tightness $\theta^*$.

2. Given the labour market tightness $\theta^*$, the Beveridge curve determines the unemployment rate $u^*$. 
Wage and labour market tightness in equilibrium:

**Job creation curve:**

\[ w = y - \frac{h}{m(\theta)}(r + q) \]

**Wage curve:**

\[ w = z + \gamma(y + h\theta - z) \]
\[ = z + \gamma(y - z) \frac{r + q + \theta m(\theta)}{r + q + \gamma \theta m(\theta)} \]

**Monotonicity** of the job creation curve and the wage curve imply exists a **unique labour market equilibrium.**
Figure 2.4: Wage and labour market tightness in equilibrium
Unemployment rate in equilibrium:

Beveridge curve:

\[ u = \frac{q}{q + \theta^* m(\theta^*)} \]

Labour market tightness line:

\[ v = \theta^* u \]

The labour market tightness line pictures all vacancy rate-unemployment rate combinations that are consistent with the equilibrium labour market tightness \( \theta^* \).
Figure 2.5: Equilibrium unemployment rate
2.7 Comparative statics

In order to determine impact of a change in an exogenous parameter on the wage $w$, the market labour tightness $\theta$ and the unemployment rate $u$ in equilibrium proceed as follows:

1. Determine the impact on the wage curve.
2. Determine the impact on the job creation curve.
   \[\Rightarrow\] The new wage curve and the new job creation curve determine the new equilibrium wage and labour market tightness.

3. Determine the impact on the Beveridge curve.
4. The change in the equilibrium labour market tightness determines the change in the labour market tightness line.
   \[\Rightarrow\] The new Beveridge curve determines together with the new labour market tightness, the new equilibrium unemployment rate.
Comparative statics for the job creation curve:

The job creation curve

\[ w = y - \frac{h}{m(\theta)} (r + q) \]

- shifts up, if productivity \( y \) increases,
- rotates downward, if the cost of creating a vacancy \( h \) increases,
- rotates downward, if the discount rate \( r \) or the job destruction rate \( q \) increase,
- rotates upward, if the matching efficiency increases \((m(\theta) \text{ increases for a given labour market tightness } \theta)\).

Intuition:

- A higher productivity, a lower discount rate or a lower job destruction rate increase the value of employing a worker. This increases vacancy creation at a given wage.
- Higher vacancy creation costs or a decrease in the matching efficiency increase the recruitment cost. This decreases vacancy creation at a given wage.
Figure 2.6: Comparative statics of the job creation curve
Comparative statics for the wage curve:

The wage curve

\[
\begin{align*}
w &= z + \gamma (y + h\theta - z) \\
   &= z + \gamma (y - z) \frac{r + q + \theta m(\theta)}{r + q + \gamma \theta m(\theta)}
\end{align*}
\]

- shifts up, if productivity \( y \) or unemployment benefits \( z \) increase,
- rotates upward, if the cost of creating a vacancy \( h \) or the matching technology increase,
- rotates downward, if the discount rate \( r \) or the job destruction rate \( q \) increases,
- shifts up and rotates upward, if the workers bargaining power \( \gamma \) increases.
**Intuition:**

- A higher productivity, a lower discount rate or a lower job destruction rate increases the total match surplus and thereby increases the wage at a given market tightness.

- Higher unemployment benefits increase the value of unemployment. This increases the worker’s outside option during the bargaining process and hence the wage at a given market tightness.

- Higher vacancy creation costs reduce the firm’s outside option during the bargaining process and thereby increase the worker’s wage at a given market tightness.

- A higher matching efficiency increases an unemployed workers matching probability $\theta m(\theta)$. This increases the worker’s outside option during the bargaining process and hence the wage at a given market tightness.

- A higher bargaining power increases a worker’s share of the match surplus and therefore increases the worker’s wage at a given market tightness.
Figure 2.6: Comparative statics of the wage curve
Overall effect on the labour market tightness:

Some parameter changes shift (rotate) both the job creation and the wage curve in the same direction.

The overall effect on the labour market tightness and wage can only be determined analytically not graphically.

Substituting the wage from the job creation and the wage curve implies:

\[ G = (1 - \gamma) (y - z) - \frac{r + q + \gamma \theta m(\theta)}{m(\theta)} h = 0 \] (26)

The implicit function theorem allows us to determine the comparative statics, i.e.

\[ \frac{d\theta}{dx} = -\frac{G_x'}{G_\theta'} \quad \text{where} \quad x \in \{y, z, \gamma, r, q, h\} . \]
The derivative $G'_\theta$ is always negative, i.e.

$$G'_\theta = -\frac{\gamma ([\theta m(\theta)]' m(\theta)) - m'(\theta) [r + q + \gamma \theta m(\theta)]}{m(\theta)^2} h < 0,$$

This implies that the equilibrium labour market tightness $\theta$

- increases with productivity $y$ and
- decreases with unemployment benefits $z$, workers’ bargaining power, the discount rate $r$, the job destruction rate $q$, the cost of vacancy creation $h$ or $\eta$ the matching efficiency.

$\implies$ The change in the equilibrium wage $w$ can be obtained from looking at the wage curve.
Comparative statics for the Beveridge curve:

The Beveridge curve

\[ u = \frac{q}{q + \theta m(\theta)} \]

- shifts outward, if the job destruction rate \( q \) increases,
- shifts inward, if the matching efficiency increases,

Intuition:

- An increase in the job destruction rate \( q \) increases the inflow into unemployment. This increases the unemployment rate \( u \) for a given labour market tightness.
- An increase in the matching efficiency increase the outflow out of unemployment. This decreases the unemployment rate \( u \) for a given labour market tightness.
Figure 2.7: Comparative statics of Beveridge curve
2.8 Out of steady state dynamics

Idea:
The Mortensen-Pissarides model has been used to model fluctuations over the business cycle:

- changes in unemployment rates,

- changes in hiring and firing rates,

- changes in wages.

So far we have only looked at changes in steady state but not at the transition dynamics.
Out of steady state Bellman equations:

Unemployed and employed workers:

\[
\begin{align*}
    rV_u &= z + \theta m(\theta) [V_e(w) - V_u] + \dot{V}_u, \\
    rV_e(w) &= w + q [V_u - V_e(w)] + \dot{V}_e(w).
\end{align*}
\]  

(27) \hspace{1cm} (28)

Open and filled vacancies:

\[
\begin{align*}
    r\Pi_v &= -h + m(\theta) [\Pi_e(w) - \Pi_v] + \dot{\Pi}_v, \\
    r\Pi_e(w) &= y - w + q [\Pi_v - \Pi_e(w)] + \dot{\Pi}_e(w).
\end{align*}
\]  

(29) \hspace{1cm} (30)

Match surplus:

\[
\begin{align*}
    S &= V_e - V_u + \Pi_e - \Pi_v, \\
    \dot{S} &= \dot{V}_e - \dot{V}_u + \dot{\Pi}_e - \dot{\Pi}_v.
\end{align*}
\]  

(31) \hspace{1cm} (32)
Adjustment assumptions:

1. Vacancies can be created without delay:
   \[ \Pi_v = 0 \Rightarrow \dot{\Pi}_v = 0. \]

2. Wages can be renegotiated at any point in time:
   \[ V_e - V_u = \gamma S \quad \text{and} \quad \Pi_e - \Pi_v = (1 - \gamma) S. \]

such that wages only change according to the changes in the market tightness \( \theta \).
Dynamics of the labour market tightness:

The free entry condition $\Pi_v = 0$ and $\dot{\Pi}_v = 0$ implies that the cost of recruiting a worker always equals the value of employing the worker, i.e.

$$\Pi_e = \frac{h}{m(\theta)}.$$

The match surplus can according to the sharing rule be written as

$$S = \frac{h}{(1 - \gamma)m(\theta)},$$

and the surplus dynamics are therefore given by

$$\dot{S} = -\frac{hm'(\theta)}{(1 - \gamma)m(\theta)^2}\dot{\theta}.$$
Dynamics of the labour market tightness:

Substituting the equations (27), (28) and (30) into the match surplus equation (31) implies a second equation for the match surplus, i.e.

\[
(r + q) S = y - z - \theta m(\theta) [V_e - V_u] + \dot{S} \\
= y - z - \theta m(\theta) \gamma S + \dot{S},
\]

where the last equality follows again from the surplus sharing rule.

Substituting \( S \) and \( \dot{S} \) implies the following first order non-linear differential equation, i.e.

\[
\frac{hm'(\theta)}{m(\theta)^2} \dot{\theta} + h \frac{r + q + \theta m(\theta) \gamma}{m(\theta)} - (1 - \gamma) (y - z) = 0
\]

In steady state the market tightness \( \theta^* \) satisfies,

\[
\frac{r + q + \gamma \theta^* m(\theta^*)}{m(\theta^*)} h - (1 - \gamma) (y - z) = 0.
\]
Dynamics of the labour market tightness:

Linearization of the non-linear differential equation around the steady state allows us to characterize the adjustment path. Linearization implies

\[ \dot{\theta} + A \theta = \theta^* \text{ with } A = \gamma \frac{m(\theta^*)^2}{m'(\theta^*)} - (r + q) < 0 \]

The general solution to this differential equation is

\[ \theta = B e^{-At} + \theta^*. \]

- The adjustment path diverges, since \( A < 0 \).
- The only possible solution therefore implies \( B = 0 \).
- This implies that the labour market tightness jumps immediately to the new value, i.e.

\[ B = 0 \implies \theta = \theta^*. \]
**Intuition for the jump in the labour market tightness:**

Since vacancies can be opened any time and wages are renegotiated any time, firms anticipate the future profits and create the number of vacancies such that the market tightness jumps right to the new steady state value.

**Dynamics of the unemployment rate:**

The unemployment rate is the only variable that cannot jump to its new steady state immediately, since finding a job takes time, i.e.

$$ u = q - [q + \theta^* m (\theta^*)] u $$

The jump of the labour market tightness to its new steady state value leads to a gradual adjustment of the unemployment rate into the new steady state.
Different types of shocks causing out of steady state dynamics:

1. Aggregate shocks:

Changes in demand or supply of goods that do not shift the Beveridge curve, like changes in productivity $y$, discount rate $r$ or workers’ bargaining power $\gamma$.

$\implies$ The economy switches from a steady state with low unemployment and high vacancy rate into an economy with high unemployment and low vacancy rate.

Policy implications:

Aggregate shocks may require policies that support aggregate demand.
Figure 2.9: Dynamics after an aggregate shock
2. Reallocation shocks:

Shocks that shift the Beveridge curve due to restructuring of production processes which cause skill mismatch and reduces the matching efficiency, or causes less stable jobs and increases the job destruction rate $q$.

$\Rightarrow$ The economy switches from a steady state with low unemployment and low vacancy rate into an economy with high unemployment and high vacancy rate.

**Policy implications:**

Reallocation shocks may require structural reforms.
Figure 2.10: Dynamics after an reallocation shock
Figure 9.2
The Beveridge curves in the United Kingdom, the United States, France, and Germany.
Sources: OECD data.
Performance of the basic Mortensen-Pissarides model:

Merz (1995) and Andolfatto (1996):
The immediate adjustments of vacancies and wage cannot describe the observed adjustment lags of the unemployment and employment rate in the US economy.

Den Haan, Ramey and Watson (2000):
Taking adjustment costs of capital and endogenous job destruction into account improves the fit of the model.

Cole and Rogerson (1999):
The calibrated version of the Mortensen-Pissarides model with endogenous job destruction and aggregate shocks fits the US data well.
Shimer-Hall discussion:

Shimer (2005) and Hall (2005):

The Mortensen-Pissarides model does not fit certain moments of the US data.

**US data**

<table>
<thead>
<tr>
<th></th>
<th>$u$</th>
<th>$v$</th>
<th>$v/u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sd$</td>
<td>0.190</td>
<td>0.202</td>
<td>0.382</td>
</tr>
<tr>
<td>$autocorr$</td>
<td>0.936</td>
<td>0.940</td>
<td>0.941</td>
</tr>
</tbody>
</table>

**Mortensen-Pissarides model with aggregate shocks in $y$:**

<table>
<thead>
<tr>
<th></th>
<th>$u$</th>
<th>$v$</th>
<th>$v/u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sd$</td>
<td>0.009</td>
<td>0.027</td>
<td>0.035</td>
</tr>
<tr>
<td>$autocorr$</td>
<td>0.939</td>
<td>0.835</td>
<td>0.878</td>
</tr>
</tbody>
</table>
Shimer-Hall discussion:

- In the U.S., standard deviation of $\theta >>$ standard deviation of $\theta$ in the Mortensen-Pissarides model.
- Mortensen-Pissarides model should be extended to allow for shocks in $y$ and $q$.
- Shock that change $y$ alter wages as well. The associated changes in profits generate only a small movements along the Beveridge curve.
- A shock to $q$ generates a positive correlation between unemployment and vacancies.

⇒ Number of papers have "fixed" the problem by recalibrating the model, changing the matching function or introducing sticky wages.
Recalibrating the model:

**Hagedorn and Manovskii (2005):**

Nothing wrong with the model it just needs to be calibrated correctly, i.e.

- Set $z = 0.95$ and $\gamma = 0.05$ in the wage equation, i.e. $w = (1 - \gamma) z + \gamma (y + h\theta)$.

- implies replacement rate $\left(\frac{z}{w}\right) \simeq 1$, which is acceptable for low skilled workers but not necessarily for high skilled workers.

- The new calibration implies that changes in $z$ generate a lot stronger labor supply response than in the data.

Changing the matching function:

**Ebrahimi and Shimer (2010):**

Introducing a stock-flow matching function improves the fit considerably.
Introducing sticky wages:

Gertler and Trigari (2006):
Wages are sticky, because of infrequent wage bargaining.

Menzio and Moen (2010):
Wages are sticky, because firms offer wage contracts that insure workers against income fluctuations.

Menzio and Shi (2011):
Wages are sticky, if matches are experience goods, since workers are less willing to change jobs and work for another employer.
2.9 Efficiency of market equilibrium

Definition:

A market equilibrium is efficient, if it maximizes aggregate social welfare in the economy.

Note, the policy that implements the aggregate social welfare maximum might not be Pareto improving, i.e. some individuals might be worse off.

Constraint efficiency:

The social planner can achieve the first best allocation, if it can allocate unemployed workers directly to vacancies, i.e. if it can circumvent matching frictions.

We assume, however, that the social planner is constrained by matching frictions and cannot circumvent them.
Positive between-group externalities:

- Workers benefit from more vacancies, since more vacancies increase the probability of finding a job.
  
  Firms might not create enough vacancies, because they do not take into account that an additional vacancy increases workers’ utility.

- Vacancies benefit from more unemployed workers (or a higher search intensity), because more unemployed workers increase the probability of finding a worker.
  
  Workers might not search efficiently, since they do not take into account that an increase in their search efficiency reduces the recruitment cost of firms.
Negative within-group congestion externalities:

- If there are more vacancies in the market, this decreases the chances of another vacancy to find an unemployed worker.
  
  Firms might create too many vacancies, since they do not take into account that any additional vacancy increases the recruitment cost of firms.

- If there are more workers (or if they search with a higher intensity), they reduce the matching probability of all other workers.
  
  Workers might search too much, since they do not take into account that their higher search intensity reduces the job finding rate of other workers.
The social welfare function:

Since individuals are risk neutral, the marginal utility of a unit of output is independent of the level of income.

Thus, the planner’s welfare criterion corresponds to the discounted value of production per capita.

Welfare each period is therefore given by

$$ W = \frac{yL + zU - hV}{N} = y(1 - u) + zu - h\theta u $$

since $\theta = \frac{v}{u}$. 
The social planner’s maximization problem:

Since the social planner is not allowed to allocate workers directly to vacancies, the social planner maximizes aggregate social welfare, i.e.

$$\max_{\theta} \int_{0}^{\infty} \left[ y (1 - u) + zu - h\theta u \right] e^{-rt} dt$$

subject to the constraint implied by matching frictions, i.e.

$$\dot{u} = q (1 - u) - \theta m (\theta) u$$
Hamiltonian:

\[ H = [y(1-u) + zu - h\theta u]e^{-rt} + \mu [q(1-u) - \theta m(\theta) u] \]

FOC:

\[ \frac{\partial H}{\partial \theta} = 0 \iff he^{-rt} = -\mu m(\theta) \left[ 1 + \frac{\theta m'(\theta)}{m(\theta)} \right] \]  \hspace{1cm} (33)

\[ \frac{\partial H}{\partial u} = \dot{\mu} \iff [-y + z - h\theta]e^{-rt} - \mu [q + \theta m(\theta)] = \dot{\mu} \]  \hspace{1cm} (34)

Transversality condition:

\[ \lim_{t \to \infty} \mu u = 0 \]
Differentiating equation (33) with respect to time $t$ implies: \[ \dot{\mu} = r\mu \]

Substituting $\mu$ using equations (33) and (34) implies the following condition for the optimal labour market tightness $\theta$, i.e.

\[
\frac{h}{m(\theta)} = \left[ y - z + h\theta \right] \frac{1 - \eta(\theta)}{r + q + \theta m(\theta)}
\]

or

\[
\frac{h}{m(\theta)} = \frac{(1 - \eta(\theta))(y - z)}{r + q + \eta(\theta)\theta m(\theta)}
\]

where $\eta(\theta)$ equals the elasticity of the matching function with respect to the unemployment rate $u$, i.e.

\[
\eta(\theta) = -\frac{\theta m'(\theta)}{m(\theta)}
\]
Hosios condition:

Compare the social planner’s solution

\[ \frac{h}{m(\theta)} = \frac{(1 - \eta(\theta))(y - z)}{r + q + \eta(\theta) \theta m(\theta)} \]

with the decentralized market solution

\[ \frac{h}{m(\theta)} = \frac{(1 - \gamma)(y - z)}{r + q + \gamma \theta m(\theta)} \]

by substituting the wage using the job creation and wage curve.

Implications:

- the decentralized market is only constrained efficient, if worker’s bargaining power equals the elasticity of the matching function (Hosios condition),

\[ \gamma = \eta(\theta) \]

- The Hosios condition is generally not satisfied in reality, thus government intervention might be necessary to achieve the constrained social optimum.
Intuition for the Hosios condition:

The Hosios condition guarantees that vacancy creation is socially efficient.

If the number of vacancies is too high, social welfare is low, because a lot of resources (in form of vacancy creation costs) are used to maintain a high market tightness. The additional gain in employment and output due to the higher job finding rate of unemployed workers falls short of the associated cost of vacancy creation.

If the number of vacancies is too low, social welfare is also low, because the low job finding rate decreases employment and therefore output. If more vacancies were created, output would increase more than the associated cost of vacancy creation.